A construction of complete-simple distributive lattices

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Introduction

In this note, we prove the following result:

Theorem

There exists an infinite complete distributive lattice K with only the two trivial complete congruence relations.

The construction

The following construction is crucial in the proof of our Theorem:

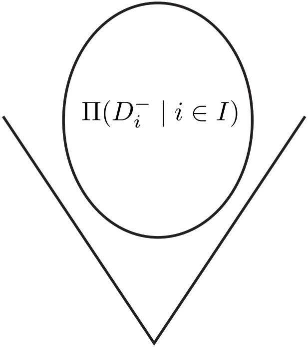
Definition

Let D_i, for $i \in I$, be complete distributive lattices satisfying condition (J). Their Π^* product is defined as follows:

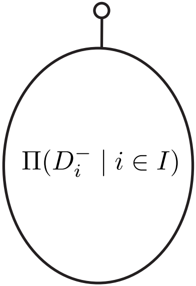
$$\Pi^*(D_i \mid i \in I) = \Pi(D_i^- \mid i \in I) + 1;$$

that is, $\Pi^*(D_i \mid i \in I)$ is $\Pi(D_i^- \mid i \in I)$ with a new unit element.

Illustrating the construction



$\Pi(D_i \mid i \in I)$



$\Pi(D_i^- \mid i \in I) + 1$

Notation

If $i \in I$ and $d \in D_i^-$, then

$$\langle \ldots, 0, \ldots, d, \ldots, 0, \ldots \rangle$$

is the element of $\Pi^*(D_i \mid i \in I)$ whose i-th component is d and all the other components are 0.
See also Ernest T. Moynahan [**?**].

The second result

Next we verify the following result:

Theorem

Let D_i, $i \in I$, be complete distributive lattices satisfying condition (J). Let Θ be a complete congruence relation on $\Pi^*(D_i \mid i \in I)$. If there exist $i \in I$ and $d \in D_i$ with $d < 1_i$ such that, for all $d \leq c < 1_i$,

$$\langle \ldots, d, \ldots, 0, \ldots \rangle \equiv \langle \ldots, c, \ldots, 0, \ldots \rangle \pmod{\Theta}, \qquad (1)$$

then $\Theta = \iota$.

Starting the proof

Since

$$\langle \dots, d, \dots, 0, \dots \rangle \equiv \langle \dots, c, \dots, 0, \dots \rangle \pmod{\Theta}, \tag{2}$$

and Θ is a complete congruence relation, it follows from condition (J) that

$$\langle \dots, d, \dots, 0, \dots \rangle \equiv \bigvee (\langle \dots, c, \dots, 0, \dots \rangle \mid d \leq c < 1) \pmod{\Theta}. \tag{3}$$

Completing the proof

Let $j \in I$, $j \neq i$, and let $a \in D_j^-$. Meeting both sides of the congruence (**??**) with $\langle \ldots, a, \ldots, 0, \ldots \rangle$, we obtain that

$$0 = \langle \ldots, a, \ldots, 0, \ldots \rangle \pmod{\Theta}, \tag{4}$$

Using the completeness of Θ and (**??**), we get:

$$0 \equiv \bigvee (\langle \ldots, a, \ldots, 0, \ldots \rangle \mid a \in D_j^-) = 1 \pmod{\Theta},$$

hence $\Theta = \iota$.

References

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